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THE BUNDLE OF PARACOMPLEX STRUCTURES

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Abstract: Under consideration are the bundle of paracomplex structures and the related problems of the existence of a paracomplex structure on a manifold. We obtain some explicit descriptions for the bundle of paracomplex structures for spheres of dimensions 2, 4, and 6. The existence is proved of a nonintegrable almost paracomplex structure on the six-dimensional sphere.

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§ 1. Introduction

An *almost paracomplex structure* on a manifold M of dimension $2n$ is a continuous field of automorphisms of the tangent spaces whose square is the identity operator and whose eigensubspaces have dimension n. The notion of almost paracomplex structure is a special case of the notion of almost product structure and is an antipode to the notion of almost complex structure on a manifold. For paracomplex and para-Kähler structures, there are results and objects analogous to the notions for almost complex structures such as integrability, the fundamental 2-form, the associated metric, etc. The most complete relevant information is presented in [1, 2]. For the bundle of complex structures, there are many known results and descriptions. But there is rather scarce information for the bundle of paracomplex structures. In this paper, we define and describe in detail the bundle of paracomplex structures. In particular, we show the relationship between the bundle of orthogonal paracomplex structures and the Grassmann bundle and describe the structure of the bundle of orthogonal paracomplex structures for even-dimensional spheres of dimensions 2, 4, and 6. Since an almost paracomplex structure on a manifold M can be viewed as a global section of the bundle of paracomplex structures on M, the question of the existence of an almost paracomplex structure on a manifold is often reduced to the problem of the existence of a global section of the bundle of paracomplex structures. Using this approach, we show that, on the four-dimensional sphere, there does not even exist a nonintegrable almost paracomplex structure, and the six-dimensional sphere admits a nonintegrable almost paracomplex structure.

Note that some almost paracomplex structures were obtained for the six-dimensional pseudo-Euclidean pseudosphere in space in [3]. In this paper, we prove the existence of nonintegrable almost paracomplex structures on the standard six-dimensional sphere in the Euclidean space.

In Section 2, we give a definition of an almost paracomplex structure and give necessary information on almost paracomplex structures. In Section 3, we give the definition of the bundle of orthogonal paracomplex structures and prove the relationship between this bundle and the Grassmann bundle. In Section 4, we describe the bundle of orthogonal paracomplex structures over the four-dimensional sphere and prove that this bundle admits no global sections. In Section 5, we prove the existence on the six-dimensional sphere in the Euclidean space of a nonintegrable almost paracomplex structure with the use of a 3-form with nontrivial radical by analogy with the way it was done in [4] for defining the notion of subtwistor and sub-Kähler structures with degenerate fundamental 2-form.

Note that, while in [3], the explicit form is given of almost paracomplex structures for six-dimensional pseudospheres, in the article we only prove the existence of a paracomplex structure on the standard six-dimensional sphere but there is no way to simply and explicitly describe this paracomplex structure.

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§ 2. Almost Paracomplex Structures

Let V be a vector space of dimension $2n$ over the field of reals. Let Φ be a nondegenerate linear operator in V such that $\Phi^2 = \mathrm{id}$, where id is the identity operator. The condition $\Phi^2 = \mathrm{id}$ implies that Φ can have two eigenvalues: ± 1. Denote by V_+ and V_- the eigenspaces corresponding to the eigenvalues 1 and -1. If $\dim(V_+) = \dim(V_-) = n$ then Φ is called a *paracomplex structure* on V. The simplest example of a paracomplex structure is given by a linear operator in V having in a fixed basis the matrix

$$\begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & -\mathrm{id}_n \end{pmatrix}.$$

If M is a smooth manifold of dimension $2n$; then for each $x \in M$ there is a paracomplex structure in the tangent space T_xM and $T_xM = V_+(x) \oplus V_-(x)$, where $V_+(x)$ and $V_-(x)$ are the eigenspaces of the paracomplex structure in T_xM.

Definition 2.1. An *almost paracomplex structure* on a real manifold M of dimension $2n$ is a continuous field Φ of automorphisms of the tangent spaces on M such that $\Phi^2 = \mathrm{id}$, while the distributions of the eigenspaces V_+ and V_- are continuous distributions of rank n on M.

Remark 2.2. An almost paracomplex structure on a manifold M is a particular case of an almost product structure. An *almost product structure* on M is a continuous field of automorphisms of the tangent spaces on M whose square is the field of identity operators and whose distributions of the eigenspaces $V_+, V_- : TM = V_+ \oplus V_-$ have arbitrary rank.

An *almost para-Hermitian structure* on a manifold M of dimension $2n$ is a pair (Φ, h), where h is a pseudo-Riemannian metric of signature (n, n) on M, while Φ is a paracomplex structure on M and $h \circ \Phi = -h$. The eigenspaces $V_+(x)$ and $V_-(x)$ of an almost para-Hermitian structure at any point $x \in M$ are isotropic subspaces in T_xM.

Remark 2.3. Using a partition of unity, we can always construct a pseudo-Riemannian metric of signature (n, n) on each paracompact manifold of dimension $2n$. If q is such a pseudo-Riemannian metric then each almost paracomplex structure Φ on M together with the pseudo-Riemannian metric $h(X, Y) = q(X, Y) - q(\Phi X, \Phi Y)$, $X, Y \in C^1(TM)$, constitutes an almost para-Hermitian structure on M.

Let M be a Riemannian manifold with Riemannian metric g. An almost paracomplex structure Φ on M is called *orthogonal* if $g \circ \Phi = g$. For an orthogonal almost complex structure, the distributions of the eigenspaces V_+ and V_- are orthogonal with respect to g. Note that if an almost paracomplex structure Φ is not orthogonal with respect to g then Φ is orthogonal with respect to the metric h; i.e., $h(X, Y) = g(X, Y) + g(\Phi X, \Phi Y)$, $X, Y \in C^1(TM)$. Thus, every almost paracomplex structure on a Riemannian manifold is always orthogonal with respect to some Riemannian metric.

The *torsion tensor of an almost paracomplex structure* Φ on a manifold M is the tensor field N of type $(2, 1)$ such that, for all vector fields X and Y on M,

$$N(X, Y) = [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] + [X, Y], \tag{1}$$

where $[\cdot, \cdot]$ stands for the Lie bracket of vector fields on M. This tensor is an analog of the Nijenhuis tensor for almost paracomplex structures.

In each coordinate neighborhood U with local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on every real manifold of dimension $2n$, there exists a local almost paracomplex structure $\Phi|_U$:

$$\Phi \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}, \quad \Phi \frac{\partial}{\partial y_l} = -\frac{\partial}{\partial y_l}, \tag{2}$$

where $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial y_l}$ are local vector fields acting on every smooth function f as the partial derivatives of f with respect to the coordinates x_k and y_l. But conversely it is not always that for an almost paracomplex structure Φ on a manifold M there exist local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ for which (2) is fulfilled. An almost paracomplex structure on a manifold M is called *integrable*, or *paracomplex*, if, for any point $x \in M$, there exists a coordinate neighborhood in which (2) is fulfilled. The following integrability criteria were obtained for almost paracomplex structures (see [1, 2]):

Theorem 2.4. *Let Φ be an almost paracomplex structure on a smooth real manifold M of dimension $2n$ and let N be the torsion tensor of Φ. Then the following are equivalent:*

(1) *Φ is an integrable almost paracomplex structure on M;*

(2) *the distributions of the eigenspaces V_+ and V_- are involutive;*

(3) *$N = 0$ on M.*

Let M and N be smooth real manifolds of the same dimension. The manifold $M \times N$ always admits an almost paracomplex structure Φ; i.e., $\Phi x = X|_{X \in C^1(TM)}$ and $\Phi X = -X|_{X \in C^1(TN)}$. Theorem 2.4 implies that this almost paracomplex structure is integrable. Thus, we conclude that there always exists an integrable almost paracomplex structure on a direct product of smooth manifolds. In particular, some integrable paracomplex structure exists on $S^n \times S^n$, $n \geq 1$, where S^n is the n-dimensional unit sphere. Conversely, we can prove the following (see [1, 2]):

Theorem 2.5. *Let M be a smooth real manifold of dimension $2n$. If M admits an integrable almost paracomplex structure then M is locally diffeomorphic to the direct product of two n-dimensional submanifolds.*

REMARK 2.6. An almost paracomplex structure is often called integrable if its distributions of eigenspaces are involutive. Theorem 2.4 implies that this definition is equivalent to the above-given definition of integrability of an almost paracomplex structure.

§ 3. The Bundle of Paracomplex Structures

In this section, we define the notion of bundle of paracomplex structures over an even-dimensional manifold and its relationship with the Grassmann bundle.

Let M be a smooth real manifold of dimension $2n$ with Riemannian metric g. For each point $x \in M$, the tangent space T_xM is a vector space of dimension $2n$ with the inner product induced by the Riemannian metric g. Denote by $\mathscr{O}_x(M)$ the set of all orthogonal paracomplex structures in T_xM. The eigenspaces $V_+(x)$: $\Phi_x|_{V_+(x)} = \mathrm{id}$ and $V_-(x)$: $\Phi_x|_{V_-(x)} = -\mathrm{id}$ are defined for every orthogonal paracomplex structure Φ_x in T_xM. Since $V_+(x)$ and $V_-(x)$ are orthogonal and $T_xM = V_+(x) \oplus V_-(x)$, the paracomplex structure Φ_x is completely determined by the choice of one of these subspaces. For every n-dimensional subspace $D(x) \subset T_xM$, there exists a unique orthogonal complement $D^\perp(x)$. With the subspace $D(x)$, we can associate the two paracomplex structures; namely, Φ_x: $\Phi_x|_{D(x)} = \mathrm{id}$, $\Phi_x|_{D^\perp(x)} = -\mathrm{id}$, and $-\Phi_x$: $-\Phi_x|_{D(x)} = -\mathrm{id}$, $-\Phi_x|_{D^\perp(x)} = \mathrm{id}$. Observe that, for n even, the paracomplex structure $\pm\Phi_x$ preserves orientation in T_xM; and for n odd, it changes orientation in T_xM. Thus, we obtain

Proposition 3.1. *Let M be a real smooth manifold of dimension $2n$. For every point $x \in M$, the space $\mathscr{O}_x(M)$ is a trivial two-sheeted covering of the n-Grassmannian in $\mathbb{R}^{2n}$.*

Denote by $\mathscr{O}(M)$ the bundle over a smooth real manifold M of even dimension for which there exists a smooth function $\pi : \mathscr{O}(M) \to M$ such that $\pi^{-1}(x) = \mathscr{O}_x(M)$ for every $x \in M$. We will refer to this bundle as the *bundle of orthogonal paracomplex structures*. Since each paracompact manifold admits a Riemannian metric (see [5, 6]), the bundle of orthogonal paracomplex structures exists for every paracompact manifold. Thus, an almost paracomplex structure on a paracompact manifold M can be defined as a global section of $\mathscr{O}(M)$. Here we take into account the result of Section 2: for every almost paracomplex structure on a Riemannian manifold, there exists a Riemannian metric with respect to which this paracomplex structure is orthogonal.

Denote by gr^k the k-Grassmannian in $\mathbb{R}^n$ and designate as $\mathrm{gr}^k(M)$ the k-Grassmann bundle over a manifold M. Proposition 3.1 yields

Corollary 3.2. *The bundle of orthogonal paracomplex structures over a Riemannian manifold M of dimension $2n$ is a trivial two-sheeted covering of $\mathrm{gr}^n(M)$.*

Since an almost paracomplex structure on a paracompact manifold M is a global section of $\mathscr{O}(M)$, Corollary 3.2 gives a criterion for the existence of an almost paracomplex structure.

Proposition 3.3. *A paracompact manifold M of dimension $2n$ admits an almost paracomplex structure if and only if M admits a global section of* $\mathrm{gr}^n(M)$.

Let $\mathrm{O}(k)$ be the orthogonal group of the Euclidean space $\mathbb{R}^k$ and let $\mathrm{U}(k)$ be the unitary group of the Hermitian space $\mathbb{C}^k$. Since $\mathrm{O}(n)$ acts transitively on the k-Grassmannian in the Euclidean space $\mathbb{R}^n$ and the isotropy subgroup of a fixed k-dimensional subspace in $\mathbb{R}^n$ consists of the block matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A \in \mathrm{O}(k)$ and $B \in \mathrm{O}(n-k)$, we conclude that $\mathrm{gr}^k \cong \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$ and $\dim(\mathrm{gr}^k) = k(n-k)$. Since, for a Riemannian manifold M of dimension $2n$, the fiber of $\mathscr{O}(M)$ is a two-sheeted covering of the n-Grassmannian in $\mathbb{R}^{2n}$, we have

Proposition 3.4. *Let $\mathscr{O}(M)$ be the bundle of orthogonal paracomplex structures over a Riemannian manifold M of dimension $2n$ with projection $\pi : \mathscr{O}(M) \to M$. Then $\pi^{-1}(x)$ is a trivial two-sheeted covering of the Grassmannian $\mathrm{gr}^n \cong \mathrm{O}(2n)/(\mathrm{O}(n) \times \mathrm{O}(n))$ and $\dim(\pi^{-1}(x)) = n^2$ for every point $x \in M$.*

A manifold M of dimension n is called *parallelizable* if there is a global n-frame on M. A parallelizable manifold always admits a Riemannian metric with respect to which a global n-frame on M is orthonormal. The existence of an almost paracomplex structure on a parallelizable manifold M is equivalent to the triviality of $\mathscr{O}(M)$.

Theorem 3.5. *A real parallelizable manifold M of dimension $2n$ admits an almost paracomplex structure if and only if the bundle $\mathscr{O}(M)$ of orthogonal paracomplex structures over M is trivial.*

Proof. Suppose that M admits an almost paracomplex structure Φ. The almost paracomplex structure Φ defines on M a pair of distributions of n-dimensional tangent spaces D_+: $\Phi|_{D_+} = \mathrm{id}$ and D_-: $\Phi|_{D_-} = -\mathrm{id}$. The distribution D_+ is a global section of $\mathrm{gr}^n(M)$. Proposition 3.4 implies that the n-Grassmannian in T_xM is diffeomorphic to $\mathrm{O}(2n)/(\mathrm{O}(n) \times \mathrm{O}(n))$ for every $x \in M$. We will assume that $\mathbb{Z}_2 = \pm 1$. Let $x \in M$ and $A \in \mathrm{O}(2n)/(\mathrm{O}(n) \times \mathrm{O}(n))$. Since M is parallelizable, on M there exists a global $2n$-frame u. The frame u defines an isomorphism $\mathbb{R}^{2n} \to T_xM$ at every point $x \in M$. Put $A(v) = u \circ A \circ u^{-1}(v)$, $v \in T_xM$. It follows that $f : f(x, A, 1) = A(D_+(x))$, $f(x, A, -1) = A(D_-(x))$ is an isomorphism of the bundles $M \times \mathrm{O}(2n)/(\mathrm{O}(n) \times \mathrm{O}(n)) \times \mathbb{Z}_2$ and $\mathscr{O}(M)$; i.e., $\mathscr{O}(M)$ is trivial.

Since each trivial bundle admits a standard global section; if $\mathscr{O}(M)$ is trivial, from Proposition 3.3 we conclude that M admits an almost paracomplex structure. □

Let M be a paracompact manifold of dimension n and let $w_k(M)$ be the kth Stiefel–Whitney class of M. The Stiefel–Whitney classes give a necessary condition for the existence of a global k-frame, $1 \le k \le n$, on a manifold (see [7]).

Proposition 3.6. *Let M be a paracompact manifold of dimension $n \ge 2$. If M admits k vector fields linearly independent at every point then $w_{n-i+1}(M) = 0$ for all $i : k \le i \le n+1$.*

Consider the two-dimensional unit sphere S^2 in the Euclidean space $\mathbb{R}^3$. A global section $s(x)$ of the 1-Grassmann bundle on S^2 induces a vector field $V(x)$ on S^2, where $V(x)$ is a unit tangent vector generating a tangent straight line $s(x)$ at x. The vector field $V(x)$ no vanishes totally on S^2. Since the second Stiefel–Whitney class $w_2(S^2)$ generates the second cohomology group $H^2(S^2, \mathbb{Z}_2) \cong \mathbb{Z}_2$, Propositions 3.3 and 3.6 imply that S^2 admits no almost paracomplex structure.

Let g_0 be the Riemannian metric on S^2 induced by the embedding of S^2 in $\mathbb{R}^3$ and let $P(2)$ be the bundle of frames on S^2 orthonormal with respect to g_0. The space $P(2)$ consists of ordered pairs (e_1, e_2), $e_1, e_2 \in C^1(T(S^2))$. Let π_1 be the projection $P(2) \to T(S^2)$; i.e., $\pi_1(e_1, e_2) = e_1$, and let π_2 be the projection of the tangent bundle $T(S^2) \to S^2$. For every $x \in S^2$ and every unit tangent vector $e_1 \in T_x(S^2)$ in $T_x(S^2)$, there exist only two unit orthogonal vectors $\pm e_2$ orthogonal to e_1. The orthonormal frame (e_1, e_2) at x defines a unique orthogonal paracomplex structure Φ_x in $T_x(S^2)$ such that $\Phi_x e_1 = e_1$ and $\Phi_x e_2 = -e_2$. Thus, the space $P(2)$ together with the projection π_1 is a two-sheeted covering of the bundle $\mathrm{gr}^1(S^2)$; and, together with the projection $\pi = \pi_2 \circ \pi_1$, is a bundle over S^2. Thus, we have

Theorem 3.7. *The bundle of orthogonal paracomplex structures over the two-dimensional sphere S^2 is isomorphic to the bundle of orthonormal frames over S^2 and admits no global section.*

REMARK 3.8. For the two-dimensional sphere S^2, the bundle $\mathscr{O}(S^2)$ is a principal bundle with structure group $\mathrm{O}(2)$. This bundle admits reduction to a subbundle with the structure group $\mathrm{SO}(2)$. This subbundle is isomorphic to $T(S^2)$. In other words, the tangent bundle over S^2 is a subbundle of $\mathscr{O}(S^2)$.

§ 4. The Bundle of Paracomplex Structures over the Four-Dimensional Sphere

Here we describe the bundle of orthogonal paracomplex structures over the four-dimensional sphere S^4. It is known that there are no almost complex structures on S^4 (see [4]), the bundle of orientation-preserving orthogonal complex structures (the twistor bundle) on S^4 is isomorphic to the complex projective space $\mathbb{C}P^3$, and the bundle of all orthogonal complex structures on S^4 is isomorphic to $\mathbb{C}P^3 \times \mathbb{Z}_2$ (see [8]).

Consider the four-dimensional unit sphere S^4 in the Euclidean space $\mathbb{R}^5$ with the standard Euclidean metric. The sphere S^4 can be identified with the set of pairs $(q, a) : |q|^2 + |a|^2 = 1$, where q is a quaternion and a is a real.

Denote the space of ordered pairs of quaternions (q_1, q_2) by $\mathbb{H}^2$. Assuming that the multiplicative group of nonzero quaternions $\mathbb{H}^*$ acts on $\mathbb{H}^2$ by homotheties, we conclude that the quaternionic projective space $\mathbb{H}P^1$ is diffeomorphic to $(\mathbb{H}^2 \setminus \{0\})/\mathbb{H}^*$.

Define the following functions on $\mathbb{H}^2 \setminus \{0\}$:

$$f(q_1, q_2) = 2\frac{q_1\bar{q}_2}{|q_1|^2 + |q_2|^2}, \quad h(q_1, q_2) = \frac{|q_1|^2 - |q_2|^2}{|q_1|^2 + |q_2|^2}.$$

Since $|f(q_1, q_2)|^2 + |h(q_1, q_2)|^2 = 1$ for every $(q_1, q_2) \in \mathbb{H}^2 \setminus \{0\}$; therefore, $\pi : \mathbb{H}^2 \setminus \{0\} \to S^4$, $\pi(q_1, q_2) = (f(q_1, q_2), h(q_1, q_2)) \in \mathbb{R}^5$, is a covering mapping $\mathbb{H}^2 \setminus \{0\} \to S^4$. Since π is invariant under multiplication by nonzero quaternions, π is a diffeomorphism $\mathbb{H}P^1 \to S^4$.

Let h_0 be the quaternionic Fubini–Study metric on $\mathbb{H}P^1$ (see [6]). Refer as an *orthogonal subcomplex structure at* $x \in \mathbb{H}P^1$ to a pair $(D(x), J(x))$, where $D(x)$ is a two-dimensional real tangent subspace in $T_x(\mathbb{H}P^1)$ and $J(x)$ is a complex structure in the vector space $D(x)$ orthogonal with respect to h_0. More detail on subcomplex structures can be found in [4]. Since a two-dimensional real vector space always admits only two orthogonal complex structures (see [5, Chapter 9]), the set of all orthogonal subcomplex structures at x can be identified with $\mathrm{gr}^2 \times \mathbb{Z}_2$, where gr^2 is the 2-Grassmannian in $\mathbb{R}^4$. Applying Corollary 3.2, we obtain

Proposition 4.1. *The bundle $\mathscr{O}(S^4)$ of orthogonal paracomplex structures over the four-dimensional sphere S^4 is isomorphic to the bundle $\mathrm{gr}^2(\mathbb{H}P^1) \times \mathbb{Z}_2$ of orthogonal subcomplex structures over $\mathbb{H}P^1$ with fiber $\mathrm{O}(4)/(\mathrm{O}(2) \times \mathrm{O}(2)) \times \mathbb{Z}_2$.*

Let us now prove that $\mathscr{O}(S^4)$ admits no global sections.

Proposition 4.2. *The bundle of orthogonal paracomplex structures over the four-dimensional sphere admits no global sections.*

PROOF. Suppose that S^4 admits an almost paracomplex structure Φ. Since every almost paracomplex structure on a Riemannian manifold is orthogonal with respect to some Riemannian metric (see Section 2), we may assume that Φ is a global section of $\mathscr{O}(S^4)$. Let D_+ and D_- be distributions of eigenspaces for the almost paracomplex structure Φ. We have $\mathrm{rank}(D_+) = \mathrm{rank}(D_-) = 2$, $T(S^4) = D_+ \oplus D_-$.

Let $p(E)$ be the complete Pontryagin class of a vector bundle E and let $p_i(E)$ be the ith Pontryagin class of E. Since D_+ and D_- are vector bundles of rank 2, $p_1(D_+) = p_1(D_-) = 0$. Using the properties of the characteristic classes for the Whitney sum of vector bundles (see [7]), we obtain

$$1 + p_1(T(S^4)) = p(T(S^4)) = p(D_+) \smile p(D_-) = (1 + p_1(D_+)) \smile (1 + p_1(D_-)) = 1,$$

whence $p_1(T(S^4)) = 0$. On the other hand, the first Pontryagin class of the four-dimensional sphere S^4 generates the fourth cohomology group $H^4(S^4, \mathbb{Z}) \cong \mathbb{Z}$, and hence cannot be zero. This proves that the bundle $\mathscr{O}(S^4)$ admits no global sections.

REMARK 4.3. Since each trivial bundle admits a global section, Proposition 4.2 implies that the bundle $\mathscr{O}(S^4)$ is nontrivial.

Propositions 3.3 and 4.2 imply

Corollary 4.4. *The four-dimensional sphere has no almost paracomplex structures.*

REMARK 4.5. Since the embedding of $\mathrm{O}(2) \times \mathrm{O}(2)$ into the group $\mathrm{O}(4)$ is not a subgroup in $\mathrm{SO}(4)$, there is no reduction of $\mathscr{O}(S^4)$ to a subbundle with fiber $\mathrm{SO}(4)/(\mathrm{O}(2) \times \mathrm{O}(2)) \times \mathbb{Z}_2$.

The embedding of $\mathrm{SO}(2)\times\mathrm{SO}(2)$ into the group $\mathrm{SO}(4)$ is a subgroup in $\mathrm{SO}(4)$. Identify the Euclidean space $\mathbb{R}^4$ with the quaternionic space $\mathbb{H}$ and the Euclidean metric in $\mathbb{R}^4$, with the metric g_0 in $\mathbb{H}$, where $g_0(x, y) = \mathrm{Re}(x\bar{y})$, $x, y \in \mathbb{H}$. Let $A(q_1, q_2)$ be the linear mapping $\mathbb{H} \to \mathbb{H}$ such that $A(q_1, q_2)x = q_1 x \bar{q}_2$, $|q_1| = |q_2| = 1$, for every $x \in \mathbb{H}$. It is easy to check that $A(q_1, q_2)$ preserves g_0; i.e., $A(q_1, q_2)$ is an orthogonal operator in $\mathbb{H}$. The set of all these orthogonal operators constitutes a connected Lie group with respect to composition which is isomorphic to $\mathrm{SO}(4)$. Note that $A(-q_1, -q_2) = A(q_1, q_2)$ implying that $\mathrm{SO}(4) \cong (S^3 \times S^3)/\pm 1$. Assume that $S^1 \times S^1$ acts on $S^3 \times S^3$ as follows:

$$(a, b)(q_1, q_2) = (aq_1, bq_2),\ (a, b) \in \mathbb{C}^2, \quad |a| = |b| = 1,\ (q_1, q_2) \in \mathbb{H}^2, \quad |q_1| = |q_2| = 1.$$

Since $\mathrm{SO}(2) \cong S^1$, $S^3/S^1 \cong \mathbb{C}P^1$, we have

$$\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2)) \cong (S^3 \times S^3)/(\pm 1(S^1 \times S^1))$$
$$\cong (S^3 \times S^3)/(S^1 \times S^1) \cong (S^3/S^1) \times (S^3/S^1) \cong \mathbb{C}P^1 \times \mathbb{C}P^1.$$

We now obtain

Proposition 4.6. *The bundle $\mathscr{O}(S^4)$ admits reduction to a subbundle with fiber $\mathbb{C}P^1 \times \mathbb{C}P^1$.*

REMARK 4.7. Since $\mathbb{C}P^1$ embeds diagonally in $\mathbb{C}P^1 \times \mathbb{C}P^1$, the bundle $\mathscr{O}(S^4)$ admits a reduction to a subbundle with fiber $\mathbb{C}P^1$. But this subbundle is not isomorphic to the twistor bundle over S^4.

§ 5. Almost Paracomplex Structures on the Six-Dimensional Sphere

It is known that the six-dimensional sphere admits a nonintegrable almost complex structure (see [5, Chapter 9]). Here we will prove that the six-dimensional sphere admits a nonintegrable almost paracomplex structure. By Remark 2.6, here we will call an almost paracomplex structure *integrable* if its distributions of eigenspaces are involutive.

Let S^6 be the six-dimensional sphere in $\mathbb{R}^7$. Identify $\mathbb{R}^7$ with a Clifford algebra over $\mathbb{R}^4$. This Clifford algebra consists of the octonions $h = \sum_{k=1}^{7} a_k i_k$, where a_k is a real, $i_k^2 = -1$, $i_k i_l = -i_l i_k$, $i_1 i_2 = i_3$, $i_1 i_4 = i_5$, $i_2 i_4 = i_6$, and $i_1 i_6 = i_7$. The conjugation of an octonion h is the octonion $\bar{h} = -h$, and the multiplication of octonions is the Clifford algebra multiplication. Let g_0 be the standard Euclidean metric in $\mathbb{R}^7$. Given octonions x and y, we have $g_0(x, y) = \mathrm{Re}(x \cdot \bar{y})$. Define the 3-form $\beta(x, y, z) = g_0(x \cdot \bar{y}, z)$ in $\mathbb{R}^7$ 3-form. Denote by G_2 the group of all orthogonal linear transformations in $\mathbb{R}^7$ that preserve the 3-form β. It is known that G_2 is a simple compact Lie group of dimension 14. We have

$$S^6 = \{h \in \mathbb{R}^7 : g_0(h, h) = 1\}.$$

The group G_2 acts transitively on S^6, and the isotropy subgroup of i_4 is isomorphic to $\mathrm{SU}(3)$ (see [5, 6]). Thus, S^6 is the homogeneous space $G_2/\mathrm{SU}(3)$.

Let $M = G/H$ be a real homogeneous Riemannian space of dimension $2n$, where G is a group of isometries acting transitively on M, while H is the isotropy subgroup of the origin point $o \in M$. An almost paracomplex structure Φ on M is called *G-invariant* if $\Phi_x \circ dg = dg \circ \Phi_o$, $x = g(o)$ for every $g \in G$, where dg is the differential of g. Each G-invariant almost paracomplex structure on M is uniquely determined by a pair of G-invariant distributions of n-dimensional subspaces D_+ and D_- on M, where $\Phi|_{D_+} = \mathrm{id}$ and $\Phi|_{D_-} = -\,\mathrm{id}$. For an orthogonal G-invariant almost paracomplex structure on M, it suffices to define only the distribution D_+ since D_- can always be chosen as the orthogonal complement to D_+. Thus, the problem of obtaining a G-invariant orthogonal almost paracomplex structure on M amounts to obtaining a G-invariant distribution of n-dimensional tangent subspaces on M.

A skew-symmetric p-form on a homogeneous Riemannian space $M = G/H$ is called *G-invariant* if $\Omega \circ dg = \Omega$ for all $g \in G$. Each G-left-invariant H-right-invariant p-form on a Lie group G induces a G-invariant p-form on the homogeneous space M (see [6]).

Definition 5.1. The *radical of a p-form Ω on a manifold M* is the distribution of tangent subspaces $\operatorname{rad}\Omega = \{X \in C^1(TM) : \mathrm{I}_X \Omega = 0\}$, where $\mathrm{I}_X \Omega$ designates the $(p-1)$-form on M obtained by replacing the first argument in the p-form Ω with the vector field X.

Theorem 5.2. *A real homogeneous Riemannian space $M = G/H$ of dimension $2n$ admits a G-invariant almost paracomplex structure if and only if M has a G-invariant n-form with radical of rank n.*

Proof. It suffices to show that there exists a G-invariant distribution of n-dimensional tangent subspaces on M if and only if M has a G-invariant n-form with radical of rank n.

Let Ω be a G-invariant n-form on M and $\operatorname{rank}(\operatorname{rad}\Omega) = n$. Let $D = \operatorname{rad}\Omega$. Show that D is a G-invariant distribution of tangent subspaces on M.

Let o be the origin point of the homogeneous space M and $v \in D(o)$. Since the n-form Ω is G-invariant, for every $g \in G$ we have

$$\mathrm{I}_{dg(v)}\,\Omega = \mathrm{I}_v(\Omega \circ dg) = \mathrm{I}_v\,\Omega = 0;$$

i.e., $dg(v) \in D(x)$ and $x = g(o)$. Since dg is an isomorphism between T_oM and T_xM, $x = g(o)$, we infer that $dg(D(o)) = D(x)$, $x = g(o)$ for any $g \in G$.

Conversely, if D is a G-invariant distribution of n-dimensional tangent subspaces on M; then, at the origin point o, there exists a collection of independent 1-forms $\alpha_1, \dots, \alpha_n$ such that

$$D(o) = \{X \in T_oM : \alpha_1(X) = 0, \dots, \alpha_n(X) = 0\}.$$

Suppose that $\Omega_o = \alpha_1 \wedge \dots \wedge \alpha_n$ and $\Omega_x = \Omega_o \circ dg^{-1}$ for any point $x = g(o)$, $g \in G$. Let $e_1, \dots, e_{2n}$ be the dual basis for T_oM; i.e., $\alpha_k(e_k) = 1$ and $\alpha_k(e_l) = 0$ for $k \neq l$. We have $e_{n+1}, \dots, e_{2n} \in D(o)$. The definition of exterior product of 1-forms and Definition 5.1 imply that $\Omega_o(e_1, \dots, e_n) = 1$ and $e_{n+1}, \dots, e_{2n}$ generate $\operatorname{rad}\Omega_o = D(o)$. Since the n-form Ω and the distribution D are G-invariant, $D(x) = \operatorname{rad}\Omega_x$ at any point $x \in M$.

Let H be a compact nontrivial Lie subgroup in a Lie group G, let $\mathfrak{g}$ be the Lie algebra of G, let $\Lambda^p(\mathfrak{g})$ be the space of all left-invariant skew-symmetric p-forms on the Lie algebra $\mathfrak{g}$, and let Ad_g, $g \in G$, be the adjoined representation of an element g in the Lie algebra $\mathfrak{g}$ (see [6]). Choose a left-invariant measure μ on H and define the linear averaging operator $S_H : \Lambda^p(\mathfrak{g}) \to \Lambda^p(\mathfrak{g})$ over the subgroup H such that, for every p-form $\Omega \in \Lambda^p(\mathfrak{g})$,

$$S_H\Omega = \frac{\int\limits_H (\mathrm{Ad}_x^*\,\Omega)\mu_x}{\int\limits_H \mu}.$$

Proposition 5.3. *Let H be a nontrivial compact connected Lie subgroup in a Lie group G and $\Omega \in \Lambda^p(\mathfrak{g})$. Then*

(1) *$S_H\Omega$ is an H-bi-invariant p-form;*

(2) *if Ω is an H-bi-invariant p-form then $S_H\Omega = \Omega$;*

(3) *if $S_H\Omega \neq 0$ then* $\operatorname{rank}(\operatorname{rad}(S_H\Omega)) = \operatorname{rank}(\operatorname{rad}\Omega)$.

PROOF. Since every connected compact Lie group is unimodular (see [6]), H has a bi-invariant measure μ; i.e., $\mathrm{Ad}_h^* \mu = \mu$ for all $h \in H$. Let $x \in H$, $h \in H$, and $y = hx$. We have

$$\mathrm{Ad}_h^*(S_H\Omega) = \frac{\int\limits_H (\mathrm{Ad}_h^*(\mathrm{Ad}_x^* \Omega))\mu_x}{\int\limits_H \mu} = \frac{\int\limits_H (\mathrm{Ad}_{hx}^* \Omega)\, \mathrm{Ad}_h^* \mu_{hx}}{\int\limits_H \mu} = \frac{\int\limits_H (\mathrm{Ad}_y^* \Omega)\mu_y}{\int\limits_H \mu} = S_H\Omega.$$

This proves property (1).

The Integral Mean Value Theorem implies that there is $h_0 \in H$ such that

$$\int\limits_H (\mathrm{Ad}_x^* \Omega)\mu_x = \mathrm{Ad}_{h_0}^* \Omega \int\limits_H \mu.$$

Hence, $S_H\Omega = \mathrm{Ad}_{h_0}^* \Omega$. If a p-form Ω is H-bi-invariant then $\mathrm{Ad}_{h_0}^* \Omega = \Omega$. If $S_H\Omega \neq 0$ then, since Ad_{h_0} is an automorphism of the Lie algebra $\mathfrak{g}$, we have $\mathrm{rad}(\mathrm{Ad}_{h_0}^* \Omega) = \mathrm{Ad}_{h_0}^{-1}(\mathrm{rad}\, \Omega)$. This proves properties (2) and (3).

The tangent space $T_x(S^6)$ at $x \in S^6$ is isomorphic to the set of imaginary octonions $y \in \mathbb{R}^7$: $g_0(y,x) = 0$. Note that $x \cdot \bar{x} = g_0(x,x) = 1$ for any $x \in S^6$ and $\bar{y} = -y$ for all $y \in \mathbb{R}^7$. Define a continuous field of linear operators $J_0(x)y$, $x \in S^6$, $y \in T_x(S^6)$ on S^6 such that $J_0(x)y = x \cdot \bar{y}$. For any $y \in T_x(S^6)$, we have

$$g_0(J_0(x)y, x) = g_0(x \cdot \bar{y}, x) = -g_0(x \cdot y, x) = -\mathrm{Re}(x \cdot y \cdot \bar{x}) = -\mathrm{Re}(x \cdot \bar{y} \cdot \bar{x}) = -g_0(J_0(x)y, x),$$

Hence, $g_0(J_0(x)y, x) = 0$, i.e., $J_0(x)y \in T_x(S^6)$. Since $T_x(S^6)$ is isomorphic to the subspace of imaginary octonions, $\overline{(x \cdot \bar{y})} = -x \cdot \bar{y}$. Furthermore,

$$J_0^2(x)y = x \cdot \overline{(x \cdot \bar{y})} = -x \cdot x \cdot \bar{y} = -(\bar{x} \cdot x) \cdot y = -y,$$

$$g_0(J_0(x)y, J_0(x)y) = \mathrm{Re}(x \cdot \bar{y} \cdot \overline{(x \cdot \bar{y})}) = (x \cdot \bar{x})(\bar{y} \cdot y) = g_0(y,y).$$

Thus, J_0 is a G_2-invariant almost complex structure on S^6 orthogonal with respect to the Euclidean metric g_0. The fundamental 2-form Θ of the almost Hermitian structure (J_0, g_0) is a G_2-invariant non-degenerate skew-symmetric 2-form on S^6, whereas $d\Theta$ is a G_2-invariant 3-form on S^6.

The Lie algebra $\mathfrak{g}_2$ of G_2 admits an H-bi-invariant inner product, where $H = \mathrm{SU}(3)$ (see [6]), and $\mathfrak{g}_2$ splits into the direct sum $\mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of $\mathrm{SU}(3)$, while $\mathfrak{m}$ is the orthogonal complement to the subalgebra $\mathfrak{h}$. The subspace $\mathfrak{m}$ is Ad_H-invariant. It follows that the set of all G_2-invariant 3-forms on S^6 is in one-to-one correspondence with the set of G_2-left-invariant H-right-invariant 3-forms on the subspace $\mathfrak{m}$. In particular, the 3-form $d\Theta$ generates an H-bi-invariant 3-form $\eta : d\Theta = \eta \circ d\pi|_{\mathfrak{m}}$ on the subspace $\mathfrak{m}$, where π is the projection $G_2 \to S^6$. Proposition 5.3 implies that $S_H\eta = \eta$.

Proposition 5.4. *The subspace* $\mathfrak{m}$ *admits an* H*-bi-invariant* 3*-form* Ω, $H = \mathrm{SU}(3)$, *such that* $S_H\Omega \neq 0$ *and* $\mathrm{rank}(\mathrm{rad}\, \Omega) = 3$.

PROOF. Let $\theta_1, \dots, \theta_6$ be a left-invariant basis for $\mathfrak{m}^*$. Then the 3-forms $\alpha_{ijk} = \theta_i \wedge \theta_j \wedge \theta_k$, $i < j < k \le 6$, constitute a left-invariant basis of $\Lambda^3(\mathfrak{m})$. Since $\eta \in \Lambda^3(\mathfrak{m})$, there exist coefficients a_{ijk}, $i < j < k \le 6$, such that

$$\eta = \sum_{i<j<k} a_{ijk}\alpha_{ijk}.$$

Since the subgroup $H = \mathrm{SU}(3)$ is a compact connected Lie group, Proposition 5.3 implies that

$$\sum_{i<j<k} a_{ijk} S_H \alpha_{ijk} = S_H\eta = \eta.$$

This implies the existence of at least one collection of the indices i, j, and k for which $S_H\alpha_{ijk} \neq 0$. Without loss of generality, we may assume that $i = 1$, $j = 2$, $k = 3$. It is easy to see that $\operatorname{rank}(\operatorname{rad}\alpha_{123}) = 3$. From Proposition 5.3 we obtain that the 3-form $\Omega = S_H\alpha_{123}$ is an H-bi-invariant 3-form on $\mathfrak{m}$ and $\operatorname{rank}(\operatorname{rad}\Omega) = 3$.

Since each H-bi-invariant 3-form on the subspace $\mathfrak{m}$ induces a G_2-invariant 3-form on S^6, Theorem 5.2 and Proposition 5.4 imply that the sphere S^6 admits a G_2-invariant almost paracomplex structure generated by the almost complex structure on S^6. Denote this almost paracomplex structure by Φ_0. In Section 2, we introduced the notion of an integrable almost paracomplex structure. Prove that the almost paracomplex structure Φ_0 is nonintegrable.

Theorem 5.5. *Let $M = G/H$ be a homogeneous Riemannian space of dimension* 6, *let G be a simple Lie group acting transitively on M, and let H be a compact connected isotropy subgroup of the origin point o. Each G-invariant 3-form on M with radical of rank* 3 *generates a nonintegrable G-invariant almost paracomplex structure on M.*

Proof. Let Ω be a G-invariant 3-form on M such that $\operatorname{rank}(\operatorname{rad}\Omega) = 3$. By Theorem 5.2, the 3-form Ω defines a G-invariant almost paracomplex structure Φ on M. Suppose that the almost paracomplex structure Φ is integrable. Then the distributions of the eigensubspaces D_+ and D_- on M are involutive.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G, let $\mathfrak{h}$ be the Lie algebra of the isotropy subgroup H, and let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{h}$ with respect to the H-bi-invariant inner product in $\mathfrak{g}$. Let $\mathfrak{M}$ be an Ad_H-invariant subspace and let $[\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m}$.

Let π be the projection $G \to M$. Since $T_oM = D_+(o) \oplus D_-(o)$ and $d\pi$ is an isomorphism $\mathfrak{m} \to T_oM$, there are subalgebras $\mathfrak{m}_+ = d\pi^{-1}D_+(o)$ and $\mathfrak{m}_- = d\pi^{-1}D_-(o)$ in $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$. By Theorem 2.5, the space M is diffeomorphic to the direct product of two subspaces M_+; i.e., $TM_+ = D_+|_{M_+}$ and $M_- : TM_- = D_-|_{M_-}$. It follows that the Lie bracket of the subalgebras $\mathfrak{m}_+$ and $\mathfrak{m}_-$ is equal to zero. We conclude that $\mathfrak{m}$ is a subalgebra in $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ implies

$$[\mathfrak{g}, \mathfrak{m}] = [\mathfrak{m}, \mathfrak{m}] \oplus [\mathfrak{h}; \mathfrak{m}] \subseteq \mathfrak{m};$$

i.e., $\mathfrak{m}$ is an ideal in $\mathfrak{g}$. Since the Lie algebra $\mathfrak{g}$ is simple, in $\mathfrak{g}$ there can be no nontrivial ideals. Thus, the almost paracomplex structure Φ cannot be integrable.

Since the group G_2 is simple, Theorem 5.5 implies that the above-obtained almost paracomplex structure Φ_0 on S^6 is nonintegrable. Moreover, the proof of Theorem 5.5 implies that every G_2-invariant paracomplex structure on S^6 is nonintegrable.

The almost paracomplex structure Φ_0 is a global section of the bundle $\mathscr{O}(S^6)$ of orthogonal paracomplex structures over S^6. Applying Corollary 3.2, we obtain

Corollary 5.6. *The bundle $\mathscr{O}(S^6)$ of orthogonal paracomplex structures over the six-dimensional sphere is isomorphic to the bundle $\operatorname{gr}^3(S^6) \times \mathbb{Z}_2$, where $\operatorname{gr}^3(S^6)$ is the* 3*-Grassmann bundle over S^6, and admits a global section.*

An *almost product structure of type* (p, q) on a real manifold M of dimension n is a continuous field ψ of automorphisms of tangent spaces such that $\psi^2 = \mathrm{id}$, $\operatorname{rank}(D_+) = p$, and $\operatorname{rank}(D_-) = q$, $p + q = n$, where D_+ is the distribution of the eigenspaces for the eigenvalue 1, D_- is the distribution of the eigenspaces for the eigenvalue -1. An almost paracomplex structure on M is an almost product of type (p, p). Using the fundamental 2-form of the above-obtained Hermitian structure on S^6, in the same way as in constructing a G_2-invariant almost paracomplex structure on S^6, we can obtain a G_2-invariant skew-symmetric 2-form Ω on S^6: $\operatorname{rank}(\operatorname{rad}\Omega) = 4$. Putting $D_+ = \operatorname{rad}\Omega$ and $D_- = (\operatorname{rad}\Omega)^\perp$, we obtain a g_2-invariant almost product structure of type $(4, 2)$ on S^6. Putting $D_+ = (\operatorname{rad}\Omega)^\perp$ and $D_- = \operatorname{rad}\Omega$, we obtain a G_2-invariant almost product structure of type $(2, 4)$ on S^6. Both these almost product structures are nonintegrable, i.e., the distributions D_+ and D_- are noninvolutive. Since every vector field on S^6 vanishes at least at one point, S^6 admits no almost product structures of type $(1, 5)$ or $(5, 1)$.

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