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INVARIANT AFFINOR AND SUB-KÄHLER STRUCTURES ON HOMOGENEOUS SPACES

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Abstract: We consider G-invariant affinor metric structures and their particular cases, sub-Kähler structures, on a homogeneous space G/H. The affinor metric structures generalize almost Kähler and almost contact metric structures to manifolds of arbitrary dimension. We consider invariant sub-Riemannian and sub-Kähler structures related to a fixed 1-form with a nontrivial radical. In addition to giving some results for homogeneous spaces of arbitrary dimension, we study these structures separately on the homogeneous spaces of dimension 4 and 5.

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1. Introduction

In the classical case, the Kähler structures (Ω, J, g) are studied on manifolds of real dimension $2n$, where g is a Hermitian metric, J is an orthogonal complex structure, and Ω is the fundamental 2-form of g. The contact metric structures (θ, Φ, g) are studied on the manifolds of dimension $2n+1$, where θ is a contact form, Φ is a degenerate operator acting on the kernel of θ as an orthogonal almost complex structure, and g is the Riemannian metric associated to the 1-form θ. The radical of Ω is zero in the Kähler case, whereas the radical of the 2-form $d\theta$ is one-dimensional in the contact case. We introduce the concept of affinor metric structure, generalizing the concept of contact metric structure for manifolds of arbitrary dimension and 1-forms with the radical of arbitrary dimension. An affinor metric structure on a real manifold of arbitrary dimension is a triple $(\alpha, \mathit{\Phi}, g)$, where α is a 1-form, $\mathit{\Phi}$ is a singular operator called an *affinor*, and g is the Riemannian metric associated to α. In the case of an affinor metric structure the 2-form $d\alpha$ may have the radical of arbitrary dimension. In addition, every affinor metric structure on a manifold of arbitrary dimension always determines natural sub-Riemannian and sub-Kähler structures.

The concept of affinor structure naturally arises when we consider a bundle P over a manifold of real dimension $2n$ with fiber $S^1 \cong U(1)$. In this case the connection form ω is a 1-form on P with values in $\mathbb{R}$, while the connection curvature form Ω is $d\omega$. A Kähler structure on a complex manifold in the case that the fundamental 2-form is exact is also a particular case of an affinor metric structure. In particular, this is so on every symplectic manifold M with $H^2(M;\mathbb{R}) = 0$. An affinor metric structure also arises on a manifold of real dimension $2n+2$ which includes a symplectic submanifold whose symplectic form is exact (see Example 3 in Section 3). Contact and almost contact metric structures on the manifolds of odd dimension are also examples of affinor metric structures.

In the general case a 1-form on a manifold M can vanish at some points of M. However, when M is a homogeneous space G/H, every G-invariant 1-form on M is either nonzero at its every point or vanishes identically. Therefore, we consider the affinor metric structures and induced sub-Kähler structures on homogeneous spaces. Since there is a natural relation between G-invariant affinor metric structures on homogeneous spaces and left-invariant affinor metric structures on Lie groups, we can use many results of [1, 2]. When the manifold is of odd dimension $2n+1$, the affinor metric structures are often called

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almost contact metric structures. The strict affinor metric structures (see Section 4) on 3-dimensional homogeneous spaces are classified in [3]. Thus, the cases of dimension 4 and 5 are the next steps in the study of affinor metric structures on homogeneous spaces.

In Section 2 we obtain the properties of G-invariant 1-forms and their radicals. In Sections 3 and 4 we introduce and study the affinor metric structures and induced sub-Kähler structures on the homogeneous spaces of arbitrary dimension. In Sections 5 and 6 we present our results for homogeneous spaces of dimension 4 and 5.

2. The Radicals of Invariant 1-Forms

Consider a Riemannian homogeneous space (M, g) of real dimension $n \geq 3$, a group G of isometries of the metric g acting transitively and faithfully on M, a fixed point o in M, and the isotropy subgroup H of this point. Assume that we are given the decomposition $\mathfrak{g} = \mathfrak{p} \oplus h$ with $\mathfrak{p} \cong T_oM$, where $\mathfrak{p}$ is an Ad_H-invariant orthogonal complement to $\mathfrak{h}$ with respect to some left-invariant Riemannian metric on G. Identify the homogeneous space M with the set of left cosets of the subgroup H. This identification determines the canonical projection $\pi : G \to M$ and $\mathfrak{h} \subseteq \ker d\pi_e$, where e is the identity element of G. Denote by $\tau : \mathfrak{p} \to T_oM$ the isomorphism resulting from the restriction of $d\pi_e$ to $\mathfrak{p}$.

Definition 2.1. The *radical* of a multilinear p-form Ω at $x \in M$, where $p \geq 2$, is the tangent subspace $\operatorname{rad}\Omega_x = \{X \in T_xM : \mathrm{I}_X\,\Omega_x = 0\}$, where $\mathrm{I}_X\,\Omega$ is the $(p-1)$-form obtained from Ω by fixing the first vector argument. If the dimension of $\operatorname{rad}\Omega_x$ is independent of the point x then Ω is called a *regular* form, while the distribution $\operatorname{rad}\Omega = \bigcup_{x \in M} \operatorname{rad}\Omega_x$ of the tangent subspaces is called the *radical* of the regular form Ω on M.

A multilinear p-form Ω on a homogeneous space M is *G-invariant* whenever $\Omega_x = \Omega_o \circ dg^{-1}$ for $g \in G$ with $g(o) = x$. Below we verify that every G-invariant multilinear form is regular; i.e., its radical is a regular distribution on M. If Ω is a G-invariant multilinear p-form and $\widehat{\Omega} = \Omega \circ \tau$ then $\operatorname{rad}\widehat{\Omega} = \tau^{-1}\operatorname{rad}\Omega \oplus \mathfrak{h}$. Call a left-invariant multilinear p-form $\widehat{\Omega}$ on a Lie group G *isotropically degenerate* whenever $\mathfrak{h} \subset \operatorname{rad}\widehat{\Omega}$. Generalizing the results for Riemannian metrics (see [2, 4]), we obtain the following theorem:

Theorem 2.2. *For every positive integer p there is a bijective correspondence between the set of G-invariant multilinear p-forms on the homogeneous space $M = G/H$ and the set of G-left-invariant H-right-invariant isotropic degenerate multilinear p-forms on the Lie group G.*

By Theorem 2.2, every G-invariant Riemannian metric g on the homogeneous space $M = G/H$ induces a G-left-invariant H-right-invariant symmetric 2-form β on G with $\operatorname{rad}\beta = \mathfrak{h}$, while the restriction of β to $\mathfrak{p}$ is an Ad_H-invariant inner product. For the Riemannian homogeneous space G/H there exists a representation of the isotropy subgroup in the matrix group $\mathrm{SO}(n)$ (see [4]).

Every G-invariant 1-form α on the Riemannian homogeneous space M induces the 1-form $\hat{\alpha} = \alpha \circ d\pi$ and the exterior 2-form $d\hat{\alpha} = d\alpha \circ d\pi$ on G. Both forms are isotropically degenerate.

Definition 2.3. A 1-form α on a manifold M is *regular* whenever its exterior differential $d\alpha$ is a regular 2-form on M. The *radical* of a regular 1-form α on a manifold M, denoted by $\operatorname{rad}\alpha$, is the radical of its exterior differential $d\alpha$.

The definition directly implies that if α is a closed 1-form then $\operatorname{rad}\alpha = TM$. If α is not a closed regular 1-form then $\operatorname{rad}\hat{\alpha} = \tau^{-1}(\operatorname{rad}\alpha \oplus \mathfrak{h})$. If the dimension of M is even and $\operatorname{rad}\hat{\alpha} = \mathfrak{h}$ then $d\alpha$ is a symplectic structure on M. If $\Omega = d\alpha$ is a G-invariant exact exterior 2-form on M then $\Omega = d(\alpha + df)$ for every G-invariant function f on M. Since the unique G-invariant function on M is $f(x) = \mathrm{const}$, the 2-form Ω uniquely determines the G-invariant 1-form α. This yields the following result:

Theorem 2.4. *For the Riemannian homogeneous space $M = G/H$ of dimension $2n$ there is a bijective correspondence between the set of G-invariant exact symplectic structures on M and the set of G-left-invariant H-right-invariant 1-forms on G with the radical equal to $\mathfrak{h}$.*

We will prove that the dimension of the radical of every G-invariant 1-form is constant on the Riemannian homogeneous space, i.e., all G-invariant 1-forms are regular.

Lemma 2.5. *If α is a G-invariant 1-form on M then* $\dim(\operatorname{rad}\alpha_x) = \text{const}$.

PROOF. Given $x \in M$, take an isometry $g \in G$ with $g(o) = x$. For every $v \in \operatorname{rad}\alpha_x$ there exists $w \in T_oM$ with $w = dg^{-1}v$. Since $\mathrm{I}_v\, d\alpha_x = 0$ and the 2-form $d\alpha$ is G-invariant, we have

$$\mathrm{I}_w\, d\alpha_o = \mathrm{I}_v\, (dg^{-1})^* d\alpha_o = \mathrm{I}_v\, d\alpha_x = 0.$$

Consequently, $w \in \operatorname{rad}\alpha_o$. We infer similarly that for every $w \in \operatorname{rad}\alpha_o$ there exists $v \in T_xM$ with $w = dg^{-1}v$ and $v \in \operatorname{rad}\alpha_x$. This yields $\operatorname{rad}\alpha_o = dg^{-1}\operatorname{rad}\alpha_x$ and $\dim(\operatorname{rad}\alpha_x) = \dim(\operatorname{rad}\alpha_o)$ for $x \in M$.

Theorem 2.6. *Consider a Riemannian homogeneous space $M = G/H$ of dimension $n \geq 3$, take a G-invariant 1-form α on M which is not closed, and put $r = \operatorname{rank}(\operatorname{rad}\alpha)$.*

(1) *If n is even then so is r, and $0 \leq r \leq n-2$.*

(2) *If n is odd then so is r, and $1 \leq r \leq n-2$.*

PROOF. Demonstration of Theorem 2.6 appeared in [1]. Note that every G-invariant 1-form on a 2-dimensional homogeneous space is either closed or has zero radical. In particular, only the trivial invariant 1-form exists on the sphere S^2. This theorem has an important corollary.

Corollary 2.7. *If α is a G-invariant 1-form on a Riemannian homogeneous space M of arbitrary dimension $n \geq 3$ which is not closed then the orthogonal complement to the radical of α is always even-dimensional.*

Take a G-invariant 1-form α on a Riemannian homogeneous space M which is not closed and the corresponding isotropic degenerate G-left-invariant H-right-invariant 1-form $\hat{\alpha}$ on the group G. It is shown in [1] that $\operatorname{rad}\hat{\alpha}$ is a subalgebra in $\mathfrak{g}$. Denote it by $\mathfrak{r}$, and the subgroup $\exp(\mathfrak{r})$ by R. We have the inclusions $\mathfrak{h} \subseteq \mathfrak{r}$ and $H \subseteq R$. The subgroup R is a *radical subgroup.* As [1] shows, the radical subgroup is a closed connected subgroup and $\operatorname{Ad}^*_R \hat{\alpha} = \hat{\alpha}$, i.e., the 1-form $\hat{\alpha}$ is Ad_R-invariant. Theorem 2.6 implies that every 1-form α on an odd-dimensional Riemannian homogeneous manifold has a nontrivial radical, and the isotropy subgroup H is a proper subgroup in the radical subgroup R. This also supplies a method for constructing homogeneous symplectic spaces. If a connected Lie group G admits a left-invariant 1-form with nontrivial radical subgroup R then the homogeneous space G/R is even-dimensional and carries an exact symplectic structure. Here we assume that R acts on G by right multiplication. Now we obtain the following result:

Theorem 2.8. *Suppose that $M = G/H$ is a Riemannian homogeneous space of dimension $2n$ and the isotropy subgroup H is a maximal connected proper subgroup in G. Then, given a G-invariant 1-form on M, either it is closed or its exterior differential determines on M an exact symplectic structure.*

Denote by $e(M)$ the Euler class of M. As [5] shows, if M admits a nowhere vanishing vector field then $e(M) \equiv 0$. A Riemannian metric g on a homogeneous space M associates to each vector field X on M the 1-form $\mathrm{I}_X\, g$. Conversely, by Riesz's Theorem (see [6]), for every 1-form α on M there exists a unique vector field X with $\alpha = \mathrm{I}_X\, g$. This establishes a bijective correspondence between vector fields and 1-forms on We infer that if $e(M) \not\equiv 0$ then each G-invariant 1-form α vanishes at some point $x_0 \in M$. However, then there exists $g \in G$ with $g^{-1}(x_0) = o$ and $\alpha_o = \alpha_{x_0} \circ dg = 0$. Consequently, $\alpha_x = 0$ for every $x \in M$. We obtain the following sufficient condition for the absence of G-invariant 1-forms on a Riemannian homogeneous space.

Theorem 2.9. *No Riemannian homogeneous space $M = G/H$ with nonzero Euler class admits nontrivial G-invariant 1-forms.*

Since every vector field on a compact closed manifold of positive Euler characteristic vanishes at some point and the Riemannian metric determines an identity of vector fields with 1-forms, we obtain the following result:

Theorem 2.10. *No compact Riemannian homogeneous space $M = G/H$ of Euler characteristic $\chi(M) > 0$ admits nonzero G-invariant 1-forms.*

Since the even-dimensional sphere S^{2n} is a Riemannian homogeneous space of strictly positive Euler characteristic for all n, this yields

Corollary 2.11. *The sphere $S^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ admits no $\mathrm{SO}(2n+1)$-invariant 1-forms.*

Theorem 2.12. *Consider a Riemannian homogeneous space $M = G/H$. If $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$, where $\mathfrak{p}$ is the Ad_H-invariant orthogonal complement to the isotropy subalgebra $\mathfrak{h}$ and Ad_H acts irreducibly on $\mathfrak{p}$; then M admits no G-invariant 1-forms with nontrivial radical.*

Proof. Suppose that α is a G-invariant 1-form on M and $\hat{\alpha}$ is the corresponding G-left-invariant H-right-invariant 1-form on G. Denote the back image of $\mathrm{rad}\,\alpha$ in $\mathfrak{p}$ by E, put $\mathfrak{r} = \mathrm{rad}\,\hat{\alpha}$, and take the radical subgroup R of the 1-form $\hat{\alpha}$. Since $\mathfrak{r}$ is a subalgebra in $\mathfrak{g}$ and $H \subseteq R$; therefore, $\mathrm{Ad}_H\,\mathfrak{r} \subseteq \mathfrak{r}$. Since $E \subset \mathfrak{p}$, it follows that $\mathrm{Ad}_H\,E \subset \mathfrak{p}$. From $E = \mathfrak{r} \cap \mathfrak{p}$ we infer that

$$\mathrm{Ad}_H\,E \subseteq \mathfrak{r} \cap \mathfrak{p} = E.$$

Hence, E is a proper Ad_H-invariant subspace of $\mathfrak{p}$. Since Ad_H acts irreducibly on $\mathfrak{p}$, we have either $E = \{0\}$ or $E = \mathfrak{p}$.

Remark 2.13. If a homogeneous space M of dimension $2n+1$ satisfies the hypotheses of Theorem 2.12 then by claim (2) of Theorem 2.6 it admits no G-invariant 1-forms that are not closed.

3. Invariant Affinor Metric Structures

Consider a Riemannian homogeneous space M of dimension $n \geq 3$ with Riemannian metric g, a group G of isometries acting on M transitively and faithfully, the isotropy subgroup H of a point o, and a G-invariant 1-form α on M which is not closed. Denote by D the orthogonal complement to $\mathrm{rad}\,\alpha$ with respect to g. Corollary 2.7 implies that D is a vector bundle of even rank on M. Refer to it as the *work bundle*. It is easy to verify that the distributions $\mathrm{rad}\,\alpha$ and D are G-invariant. The restriction of $d\alpha$ to the work bundle D is a nondegenerate exterior 2-form.

Definition 3.1. Refer as an *affinor associated to* the 1-form α to a continuous field Φ of endomorphisms of the tangent spaces to M so that

$$g(\Phi X, \Phi Y) = g(X, Y) \quad \text{for all } X, Y \in C^\infty(D),$$
$$d\alpha(X, Y) = g(\Phi X, Y) \quad \text{for all } X, Y \in C^\infty(TM).$$

The definition directly implies the following properties of the affinor Φ:
(1) $\mathrm{rad}\,\alpha = \ker \Phi$;
(2) $\Phi^2|_D = -\mathrm{id}$;
(3) $\Phi^* d\alpha = d\alpha$;
(4) $g(X, Y) = d\alpha(X, \Phi Y)$ for all $X, Y \in C^\infty(D)$;
(5) if $x = g(o)$ then $\Phi_x = dg \circ \Phi_o \circ dg^{-1}$.
Refer as an *affinor metric structure* on a manifold M to a triple (α, Φ, g), where α is a regular 1-form on M with nontrivial radical, g is a Riemannian metric on M, and Φ is an affinor associated to α. Denote by $d\alpha_\Phi$ the restriction of the symmetric 2-form $d\alpha(X, \Phi Y)$ to the work bundle D. In general D can be a holonomic or a nonholonomic distribution on M. When D is totally nonholonomic, (4) implies that the pair $(D, d\alpha_\Phi)$ determines a sub-Riemannian structure on M, called an *affinor sub-Riemannian structure*. The following yields a condition for D to be a nonholonomic distribution.

Theorem 3.2. *If α is a G-invariant 1-form with nontrivial radical on a Riemannian homogeneous space $M = G/H$ and the work bundle D on M satisfies $D \subseteq \ker \alpha$ then the distribution D is nonholonomic.*

Proof. Using the invariant definition of exterior differential of a 1-form (see [7]), for all vector fields $X, Y \in C^\infty(D)$ we obtain

$$d\alpha(X, Y) = \frac{1}{2}\left(X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])\right) = -\frac{1}{2}\,\alpha([X, Y]),$$

where $[X,Y]$ is the Lie bracket of two vector fields X and Y. Since the exterior 2-form $d\alpha$ is nondegenerate on D, there exist vector fields $X', Y' \in D$ with $[X', Y'] \notin \ker\alpha$. Consequently, $[X', Y'] \notin D$, and D is not involutive. Frobenius's Theorem implies that D is a nonholonomic distribution on M.

The following space is an example of a homogeneous space admitting an affinor metric structure with holonomic work bundle.

EXAMPLE 1. Take $M = G \times T^n$, where G is a symplectic Lie group and T^n is the n-dimensional flat torus, and a left-invariant exact symplectic structure Ω on G. Then G admits a left-invariant 1-form α with $d\alpha = \Omega$. Extend α to T^n by putting $\alpha(X) = 0$ for every vector field X on T^n. This yields $\operatorname{rad}\alpha = T(T^n)$. Take a left-invariant almost complex structure J on G with $J^*\Omega = \Omega$ preserving the orientation of G and the standard Euclidean metric g_0 on the torus T^n. Define a Riemannian metric g on M as

$$g(X,Y) = \begin{cases} 0 & \text{for } X \in \mathfrak{g},\ Y \in T(T^n), \\ \Omega(X, JY) & \text{for } X \in \mathfrak{g},\ Y \in \mathfrak{g}, \\ g_0(X,Y) & \text{for } X \in T(T^n),\ Y \in T(T^n), \end{cases}$$

where $\mathfrak{g}$ is the Lie algebra of the Lie group G. It is obvious that the field of endomorphisms Φ with

$$\Phi X = \begin{cases} JX & \text{for } X \in \mathfrak{g}, \\ 0 & \text{for } X \in T(T^n) \end{cases}$$

is an affinor associated to the 1-form α. We obtain a left-invariant affinor metric structure (α, Φ, g) on M with the work bundle $D = (\mathfrak{g}, 0)$. Thus, the distribution D is holonomic.

Let us present more examples of natural affinor metric structures on homogeneous spaces.

EXAMPLE 2. Take a Riemannian homogeneous space $M = G/H$ of dimension $2n+1$ with Riemannian metric g. Every G-invariant 1-form α together with an affinor Φ determines on M the almost contact metric structure (α, Φ, g) (see [3, 8]). In this case the work bundle D lies in $\ker\alpha$, and by Theorem 3.2 it is a nonholonomic distribution on M. In the case that the distribution D is totally nonholonomic, which always holds for strictly contact metric structures, $(D, d\alpha_\Phi)$ is a G-invariant affinor sub-Riemannian structure on M.

EXAMPLE 3. Take a Riemannian homogeneous space $M = G/H$ of dimension $2n+2$ satisfying $H^2(M;\mathbb{R}) = 0$, a Riemannian metric g on M, and a degenerate G-invariant exterior 2-form Ω on M. Take a maximal symplectic submanifold Q in M of dimension $2k$; we obtain it by acting with some connected subgroup $K \subset G$ on the source point o. Since $H^2(M;\mathbb{R}) = 0$, the manifold M admits a G-invariant 1-form α with $d\alpha = \Omega$, which is a nondegenerate exterior 2-form on Q. Furthermore, $\operatorname{rad}\Omega = \operatorname{rad}\alpha$ and $\operatorname{rank}(\operatorname{rad}\alpha) = 2(n-k+1)$, while the restriction of the work bundle D to Q coincides with TQ. Take a complex structure J on the fibers of the work bundle D with $J^*\Omega = \Omega$ on assuming that J preserves the orientation of the fibers. Then the field of endomorphisms Φ of the tangent spaces on M,

$$\Phi X = \begin{cases} JX & \text{for } X \in D, \\ 0 & \text{for } X \in \operatorname{rad}\alpha \end{cases}$$

is an affinor associated to α. This yields the G-invariant affinor metric structure (α, Φ, g) on M. Since the restriction of Φ to Q is an almost complex structure on Q, it follows that Q is an almost Kähler submanifold of M with almost Hermitian metric $d\alpha_\Phi$. The almost complex structure Φ on Q determines the complex structure $\widehat{\Phi} = \tau^{-1} \circ \Phi_o \circ \tau$ on the subspace $\mathfrak{k} = \tau^{-1} D_o \subset \mathfrak{p}$. Take the linear operator ad_X on $\mathfrak{g}$ so that $\operatorname{ad}_X Y = [X,Y]$ for all $X, Y \in \mathfrak{g}$. If $\operatorname{ad}_{\widehat{\Phi}X} = \operatorname{ad}_X \circ \widehat{\Phi}$ for $X \in \mathfrak{k}$ then the almost complex structure Φ on Q is integrable (see [7, Chapter 9]) and Q is a Kähler submanifold of M.

Since the Euler characteristic of an odd-dimensional manifold always vanishes, Theorems 2.6 and 2.12 imply the corollary:

Corollary 3.3. *If $M = G/H$ is a Riemannian homogeneous space of dimension $2n+1$ and Ad_H acts irreducibly on the orthogonal complement to the isotropy subalgebra then M admits no G-invariant affinor metric structures.*

Since the n-dimensional sphere S^n is the reductive Riemannian homogeneous space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and the group $\mathrm{SO}(n)$ acts irreducibly on the orthogonal complement to the isotropy subalgebra with respect to the Killing form, we obtain the following result:

Theorem 3.4. *The n-dimensional sphere S^n with $n \geq 2$ admits no $\mathrm{SO}(n+1)$-invariant affinor metric structures with nontrivial radical.*

Take a G-invariant affinor metric structure (α, Φ, g) on a homogeneous space M. By Riesz's Theorem (see [6]), for every vector field X on M there is a vector field ξ with $\alpha(X) = g(\xi, X)$ on M, called the *characteristic vector field*. We have

$$\alpha(\xi) = g(\xi, \xi), \quad \alpha(X) = d\alpha(\xi, \Phi X).$$

Lemma 3.5. *The characteristic vector field ξ of a G-invariant affinor metric structure possesses the properties:*

(1) *If $x = g(o)$ for $g \in G$ then $dg^{-1}\xi_x = \xi_o$;*

(2) *The vector field ξ is of constant length;*

(3) *For every vector field X on M we have $d\alpha(\xi, X) = -\frac{1}{2}\,\alpha([\xi, X])$.*

Proof. (1) follows from the G-invariance of α and g. For every point $x \in M$ there is an isometry $g \in G$ with $x = g(o)$. We obtain

$$g_x(\xi, \xi) = g_o(dg^{-1}\xi, dg^{-1}\xi) = g_o(\xi, \xi) = \mathrm{const}.$$

Therefore, the length of the vector field ξ is independent of x. This yields (2).

Since $d\alpha(\xi, \xi) = 0$ and $TM = \mathbb{R}\,\xi \oplus \ker \alpha$, it suffices to justify (3) for an arbitrary vector field $X \in \ker \alpha$. Property (2) implies that $X(\alpha(\xi)) = X(g(\xi, \xi)) = 0$. Hence,

$$d\alpha(\xi, X) = \frac{1}{2}\left(\xi(\alpha(X)) - X(\alpha(\xi)) - \alpha([\xi, X])\right) = -\frac{1}{2}\,\alpha([\xi, X]).$$

When the characteristic vector field ξ is a Killing vector field, the affinor metric structure (α, Φ, g) is called a *K-affinor metric structure*. Some important properties of K-affinor metric structures were found in [1]. In the general case the characteristic vector field need not belong to the radical of α.

Definition 3.6. An affinor metric structure (α, Φ, g) is *strict* whenever its characteristic vector field belongs to $\mathrm{rad}\,\alpha$.

Theorem 3.7. *Consider a Riemannian homogeneous space $M = G/H$ and a G-invariant affinor metric structure (α, Φ, g) on M with characteristic vector field ξ. The following are equivalent:*

(1) *(α, Φ, g) is a strict affinor metric structure;*

(2) *$\mathrm{L}_\xi\,\alpha = 0$, where L_ξ is the Lie derivative in the direction of ξ;*

(3) *ξ is a geodesic vector field, i.e., $\nabla_\xi \xi = 0$, where ∇ is the Levi-Civita connection of the Riemannian metric g.*

Proof. Item (2) of Lemma 3.5 implies that $\alpha(\xi) = g(\xi, \xi) = \mathrm{const}$. Applying the expression for the Lie derivative of an exterior p-form in the direction of a vector field X, namely, $\mathrm{L}_X = d\,\mathrm{I}_X + \mathrm{I}_X\,d$ (see [7, Vol. 1]), we obtain

$$\mathrm{L}_\xi\,\alpha = d\alpha(\xi) + \mathrm{I}_\xi\,d\alpha = \mathrm{I}_\xi\,d\alpha.$$

This implies that (1) and (2) are equivalent.

Take the Levi-Civita connection ∇ of the metric G. For all vector fields X, Y, and Z on M we have

$$[X, Y] = \nabla_X Y - \nabla_Y X, \tag{3.1}$$

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{3.2}$$

By (3.2) we see that $g(\nabla_X \xi, \xi) = 0$ for every vector field X on M. In particular, $g(\nabla_\xi \xi, \xi) = 0$. Using item (3) of Lemma 3.5, as well as (3.1) and (3.2), for every $X \in \ker \alpha$ we obtain

$$0 = g(\nabla_\xi \xi, X) + g(\xi, \nabla_\xi X) = g(\nabla_\xi \xi, X) + g(\xi, \nabla_X \xi)$$
$$+ g(\xi, [\xi, X]) = g(\nabla_\xi \xi, X) + \alpha([\xi, X]) = g(\nabla_\xi \xi, X) - 2\, d\alpha(\xi, X).$$

Therefore, $g(\nabla_\xi \xi, Y) = 2\, d\alpha(\xi, Y)$ for every vector field Y on M. This proves that (1) and (3) are equivalent.

The equivalence of (2) and (3) follows by transitivity from (2) $\leftrightarrow$ (1) $\leftrightarrow$ (3).

REMARK 3.8. If g is a fixed invariant Riemannian metric on a homogeneous space M then by Theorem 3.7 the classification of all strict invariant affinor metric structures (α, Φ, g) reduces to the classification of geodesic vector fields on M and complex structures on the fibers of the work bundle D preserving the restriction of the 2-form $d\alpha$ to D and the orientation of the fibers of the work bundle. The classification of invariant Riemannian metrics on homogeneous spaces reduces to decomposing the orthogonal complement of the isotropy subalgebra $\mathfrak{h}$ as a sum of Ad_H-irreducible components (see [4]).

4. Invariant Sub-Kähler Structures

Consider a Riemannian homogeneous space M of dimension $n \geq 3$, a Riemannian metric g on M, a group G of isometries acting on M transitively and faithfully, and the isotropy subgroup H of a source point o. Assume that the Lie algebra $\mathfrak{g}$ has decomposition $\mathfrak{p} \oplus \mathfrak{h}$, where $\mathfrak{p}$ is the Ad_H-invariant orthogonal complement to the isotropy subalgebra $\mathfrak{h}$.

DEFINITION 4.1. A *sub-Kähler structure* on a homogeneous space M is a quadruple (Q, D, J, h) consisting of a holonomic distribution D of tangent subspaces on M, an integral submanifold Q of M passing through the source point O with $D|_Q = TQ$, an inner product h on D, and a complex structure J on Q with $J^* h = h$, so that h is a Kähler metric on Q.

Example 3 in Section 3 shows that we can define a sub-Kähler structure on a Riemannian homogeneous space using an affinor metric structure. However, we can also define a sub-Kähler structure separately.

EXAMPLE 1. Take $M = G \times S^1$, where G is a nonunimodular connected Lie group of dimension $n \geq 3$ and $S^1 = \{e^{it} : t \in [0, 2\pi]\}$. Take a left-invariant vector field Z on S^1 and put $[X, Y] = [X, Y]_G$ for all $X, Y \in \mathfrak{g}$ and $[X, Z] = \mathrm{tr}[\mathrm{ad}_X] Z$ for every $X \in \mathfrak{g}$. Denote by $\mathfrak{u}$ the unimodular kernel in $\mathfrak{g}$; it is an ideal of $\mathfrak{g}$. Then $\mathfrak{g}$ contains a vector ξ with $\mathrm{tr}[\mathrm{ad}_\xi] = 1$. Since $[\xi, Z] = Z$, the vectors ξ and Z generate a 2-dimensional subalgebra D. The group operation on M is

$$(g_1, h_1)\dot{(}g_2, h_2) = (g_1 g_2, h_1 \exp(\det[\mathrm{Ad}_{g_1}] \log(h_2))).$$

Define the left-invariant 1-form α on M by putting $\alpha(Z) = 1$ and $\alpha(X) = 0$ for every $X \in \mathfrak{g}$. We have

$$d\alpha(\xi, Z) = -\frac{1}{2}\, \alpha([\xi, Z]) = -\frac{1}{2}.$$

The subalgebra D generates the connected subgroup $Q = \exp(D)$. There is a unique orientation-preserving left-invariant complex structure J on D with $J^* d\alpha = d\alpha$. Since every arbitrary complex structure on a 2-dimensional subalgebra is a complex structure on the corresponding subgroup (see [7, Vol. 2]), we infer that Q is a complex subgroup in M. Defining the Kähler metric h on Q as

$$h(X, Y) = d\alpha(X, JY) \quad \text{for all } X, Y \in D,$$

we obtain the sub-Kähler structure (Q, D, J, h).

An affinor metric structure (α, Φ, g) on M with work bundle D is *sub-Kähler* whenever M includes a Kähler submanifold Q passing through the source point o so that $D|_Q = TQ$, the restriction of the affinor Φ to Q is the complex structure on Q, and $d\alpha_\Phi$ is the Kähler metric on Q. The problem is to find out when an affinor metric structure on M determines a sub-Kähler structure on M.

Theorem 4.2. *Consider a Riemannian homogeneous space $M = G/H$, an Ad_H-invariant complement $\mathfrak{p}$ to the isotropy subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$, and a G-invariant affinor metric structure (α, Φ, g) on M with work bundle D. Suppose that one of the following holds:*

(1) $D \subset \ker \alpha$*;*

(2) *the subspace $\mathfrak{p}$ includes no complex vector subspaces;*

(3) (α, Φ, g) *is a strict affinor metric structure.*

Then the affinor metric structure (α, Φ, g) is not sub-Kähler.

PROOF. Assume that M includes a Kähler submanifold Q passing through the source point o with $TQ = D|_Q$, the restriction of Φ to Q is the complex structure on Q, and the restriction of $d\alpha_\Phi$ to Q is the Kähler metric. If (1) holds then by Theorem 3.2 the work bundle D is a nonholonomic distribution on M. For every point $x \in M$ there is $g \in G$ with $g(o) = x$. Since the 1-form α is G-invariant and $\mathrm{rad}\, \alpha_o$ is orthogonal to $T_oQ = D_o$, it follows that $\mathrm{rad}\, \alpha_x$ is orthogonal to $T_x g(Q) = D_x$; i.e., $g(Q)$ is an integral submanifold passing through x. We infer that the distribution D is holonomic and arrive at a contradiction.

Take a linear isomorphism $\tau \colon \mathfrak{p} \mapsto T_oM$. Since the restriction of Φ to Q is a complex structure on Q and $T_oQ = D_o$, it follows that $\tau^{-1}D_o$ is a complex subspace of $\mathfrak{p}$. If (2) holds then we also obtain a contradiction.

Take the characteristic vector field ξ of the affinor metric structure (α, Φ, g). If (3) holds then ξ is orthogonal to $\ker \alpha$ and D. Since $\mathrm{rank}(\ker \alpha) \geq \mathrm{rank}(D)$, it follows that $D \subseteq \ker \alpha$. Theorem 3.2 implies that D is a nonholonomic distribution on M. In the same way as for condition (1), we infer that D is a holonomic distribution and again arrive at a contradiction.

The concept of normal affinor metric structure on a Lie group was introduced in [1]. A left-invariant affinor metric structure (α, Φ, g) on a Lie group G is *normal* whenever

$$\mathrm{ad}_{\Phi X} = \mathrm{ad}_X \circ \Phi$$

for every vector field X in the work bundle D. Denote by $\mathfrak{k}$ the work bundle of the normal affinor G-left-invariant H-right-invariant metric structure $(\hat{\alpha}, \widehat{\Phi}, \hat{g})$ on G. Since $\mathfrak{k}$ is orthogonal to $\mathrm{rad}\, \hat{\alpha}$ and $\mathfrak{p}$ is orthogonal to $\mathfrak{h} \subseteq \mathrm{rad}\, \hat{\alpha}$, it follows that $\mathfrak{k} \subseteq \mathfrak{p}$. If $\mathfrak{k}$ is a subalgebra then Corollary 2.7 shows that $\mathfrak{k}$ is even-dimensional and generates the subgroup $K = \exp(\mathfrak{k})$ of G. The restriction of the affinor $\widehat{\Phi}$ to K is a G-left-invariant H-right-invariant complex structure on K (see [7, Chapter 9]). Since the restriction of the metric $\hat{g}$ to K has the fundamental 2-form $d\hat{\alpha}$ and this exterior 2-form is nondegenerate on $\mathfrak{k}$, the subgroup K is an integral Kähler submanifold for the work bundle $\mathfrak{k}$. This bundle determines on G the G-invariant work bundle D on the homogeneous space M for the G-invariant affinor metric structure on M generated by the affinor metric structure $(\hat{\alpha}, \widehat{\Phi}, \hat{g})$ on G. Consequently, the normal affinor metric structure $(\hat{\alpha}, \widehat{\Phi}, \hat{g})$ on G generates the G-invariant affinor metric structure on M with the integral Kähler submanifold $Q = K(o)$ passing through the source point o with $T_oQ = D_o$. Thus, we obtain the following result:

Theorem 4.3. *Consider a Riemannian homogeneous space $M = G/H$ of real dimension $n \geq 3$. A G-left-invariant H-right-invariant normal affinor metric structure on the Lie group G with work bundle $\mathfrak{k}$ generates a G-invariant sub-Kähler affinor structure on M if and only if $\mathfrak{k}$ is an involutive distribution in the orthogonal complement to the isotropy subalgebra $\mathfrak{h}$.*

A complex structure J on a subgroup $Q \subset G$ is *associated* to an affinor Φ on the Lie group G whenever the restriction of Φ to Q coincides with the complex structure J on Q, i.e., it is an integrable almost complex structure on Q. Actually, the condition of normality of the affinor metric structure in Theorem 4.3 is necessary only in order for the work bundle to generate a subgroup with complex rather than almost complex structure. Thus, we obtain a stronger statement:

Theorem 4.4. *Consider a Riemannian homogeneous space $M = G/H$ of real dimension $n \geq 3$. An arbitrary G-left-invariant H-right-invariant affinor metric structure (α, Φ, g) on the group G with* *work bundle $\mathfrak{k}$ determines a G-invariant sub-Kähler affinor structure on M if and only if $\mathfrak{k}$ is the Lie algebra of a subgroup K of G endowed with the complex structure associated to Φ.*

REMARK 4.5. The subgroup K in Theorem 4.4 is transversal to the isotropy subgroup and its intersection with the isotropy subgroup is a discrete subgroup in G.

5. Invariant Affinor Metric Structures on the Homogeneous Spaces of Dimension 4

In this section we obtain a full description for the invariant affinor metric structures on the 4-dimensional Riemannian homogeneous spaces.

Consider a real Riemannian homogeneous space M of dimension 4 with a group G of isometries acting on M transitively and faithfully, the isotropy subgroup H of the source point o and the Ad_H-invariant orthogonal complement $\mathfrak{p} \subset \mathfrak{g}$ to the isotropy subalgebra $\mathfrak{h}$ with respect to some left-invariant Riemannian metric on G. Denote the Euler characteristic of M by $\chi(M)$.

The following classification of 4-dimensional Riemannian homogeneous spaces is given in [9].

Theorem 5.1. *Every simply-connected Riemannian homogeneous space of dimension 4 is isometric either to a Lie group with a left-invariant Riemannian metric or to one of the symmetric spaces:*

$$S^4, \quad S^2 \times S^2, \quad S^3 \times \mathbb{R}, \quad S^2 \times \mathbb{R}^2, \quad \mathbb{C}P^2.$$

By Theorem 5.1, for the Riemannian homogeneous spaces of dimension 4 we must consider the two cases: the symmetric 4-dimensional spaces not isometric to any Lie group and the 4-dimensional Lie groups. Complete classification of the Lie groups of dimension 4 is given in [10].

5.1. The symmetric Riemannian spaces of dimension 4. Since the 3-dimensional sphere S^3 is diffeomorphic to the group $\mathrm{SU}(2)$, we can treat the symmetric space $S^3 \times \mathbb{R}$ as a Lie group. We consider the case of Lie groups separately below. We have

$$\chi(S^4) = 2, \quad \chi(S^2 \times S^2) = 4, \quad \chi(\mathbb{C}P^2) = 3.$$

By Theorem 2.10, the symmetric spaces S^4, $S^2 \times S^2$, and $\mathbb{C}P^2$ admit no invariant affinor metric structures.

Assume that $M = S^2 \times \mathbb{R}^2$. If G is a group of isometries of M then $G = G_1 \times G_2$, where G_1 is a group of isometries of S^2 and G_2 is a group of isometries of $\mathbb{R}^2$. Since the restriction of every G-invariant 1-form to S^2 is a G_1-invariant 1-form on S^2, while S^2 admits no nonzero invariant 1-forms, each nonzero G-invariant 1-form on M is a G_2-invariant 1-form on $\mathbb{R}^2$. We can express the Euclidean space $\mathbb{R}^2$ as a homogeneous space only in the two ways: it is either the additive abelian group A_2 or the space $\mathrm{E}(2)/\mathrm{SO}(2)$, where $\mathrm{E}(2)$ is the group of motions of the Euclidean plane $\mathbb{R}^2$. If $G_2 \cong \mathrm{E}(2)$ then the unique $\mathrm{E}(2)$-invariant 1-form on $\mathbb{R}^2$ is the 1-form $\alpha \equiv 0$. If $G_2 = A_2$ then every G_2-invariant 1-form α on $\mathbb{R}^2$ is

$$\alpha = a dx + b dy, \quad a, b = \mathrm{const},$$

whence $d\alpha = 0$. Consequently, every G-invariant 1-form on $S^2 \times \mathbb{R}^2$ is either closed or vanishes identically. This yields the following result:

Theorem 5.2. *If a simply-connected Riemannian homogeneous space $M = G/H$ of dimension 4 is not isometric to a Lie group with a left-invariant metric then M admits no G-invariant affinor metric structures.*

5.2. The space $\mathbb{C}P^2$. We can consider the complex projective plane $\mathbb{C}P^2$ as the homogeneous space $\mathrm{SU}(3)/\mathrm{U}(2)$. By Theorem 5.2, it admits no $\mathrm{SU}(3)$-invariant affinor metric structures. Let us study the question of existence of noninvariant affinor metric structures on $\mathbb{C}P^2$. By claim (1) of Theorem 2.6, for every affinor metric structure on $\mathbb{C}P^2$ the radical is of rank 2 or rank 0.

Put $|Z|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 \neq 0$ for $(z_1, z_2, z_3) \in \mathbb{C}^3$. Introduce on $\mathbb{C}P^2$ the 1-form $\alpha = \bar{\partial} \log |Z|^2$. We have

$$d\alpha = \partial\bar{\partial} \log |Z|^2.$$

It is easy to verify that the exterior 2-form $d\alpha$ is nondegenerate. Take the complex structure J on $\mathbb{C}P^2$ induced by the multiplication by the imaginary unit. Then $J^*d\alpha = d\alpha$ and the Kähler metric h : $h(X,Y) = d\alpha(X, JY)$ is defined for all vector fields X and Y. This yields the noninvariant affinor metric structure (α, J, h) with rank 0 radical.

Denote by $p_1(E)$ the first Pontryagin class of a vector bundle E. If $\mathbb{C}P^2$ admits a 1-form with rank 2 radical and work bundle D then the tangent bundle is the Whitney sum of the subbundles D and $\operatorname{rad}\alpha$ of rank 2. We have

$$p_1(\mathbb{C}P^2) = p_1(D) \wedge p_1(\operatorname{rad}\alpha) = 0.$$

However, this contradicts the property that $p_1(\mathbb{C}P^2) \neq 0$. Consequently, $\mathbb{C}P^2$ cannot admit an affinor metric structure with rank 2 radical. Finally, we obtain the following result:

Theorem 5.3. *The Riemannian homogeneous space $\mathbb{C}P^2$ admits no affinor metric structure with rank 2 radical, but it admits a noninvariant affinor metric structure with rank 0 radical.*

5.3. The Lie groups of dimension 4. Consider a Lie group G of dimension 4 with a left-invariant 1-form α which is not closed and a left-invariant Riemannian metric β. If $\operatorname{rank}(\operatorname{rad}\alpha) = 0$ then there exists a basis e_1, e_2, e_3, e_4 of left-invariant vector fields in which $d\alpha$ becomes

$$d\alpha = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*. \tag{5.1}$$

If $\operatorname{rank}(\operatorname{rad}\alpha) = 2$ then there exists a basis e_1, e_2, e_3, e_4 of left-invariant vector fields in which $d\alpha$ becomes

$$d\alpha = e_1^* \wedge e_2^*. \tag{5.2}$$

The basis e_1, e_2, e_3, e_4 is called a *canonical basis*. As shown in [10], every left-invariant almost complex structure on G positively associated to the exterior 2-form (5.1) becomes

$$\begin{bmatrix} a & b & 0 & 0 \\ -\frac{a^2+1}{b} & -a & 0 & 0 \\ 0 & 0 & r & s \\ 0 & 0 & -\frac{r^2+1}{s} & -r \end{bmatrix}, \quad b<0, \ s<0,$$

in a canonical basis.

Similarly, every left-invariant affinor positively associated to the exterior 2-form (5.2) becomes

$$\begin{bmatrix} a & b & 0 & 0 \\ -\frac{a^2+1}{b} & -a & 0 & 0 \\ 0 & \ldots\ldots\ldots & & 0 \\ 0 & \ldots\ldots\ldots & & 0 \end{bmatrix}, \quad b<0,$$

in a canonical basis.

Put $z_1 = a + bi$ and $z_2 = r + si$. Take $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and $\mathbb{C}_-^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 < 0, \operatorname{Im} z_2 < 0\}$. We find that $\mathbb{C}_-^2$ parametrizes the set of all left-invariant affinors and left-invariant metrics associated to a 1-form α with $\operatorname{rank}(\operatorname{rad}\alpha) = 0$; also, $\mathbb{C}_-$ parametrizes the set of all left-invariant affinors associated to a 1-form α with $\operatorname{rank}(\operatorname{rad}\alpha) = 2$. Denote by $b_1(G)$ the first Betti number of G. Then the set of all left-invariant 1-forms on the 4-dimensional group which are not closed is parametrized by the points of $\mathbb{R}^4 \setminus \mathbb{R}^{b_1(G)}$. By the classification of 4-dimensional Lie algebras, there is no semisimple 4-dimensional Lie group (see [10]). Consequently, each 4-dimensional Lie group admits at least one nonzero left-invariant closed 1-form. If G is noncommutative then $b_1(G) \geq 1$, the set $\mathbb{R}^4 \setminus \mathbb{R}^{b_1(G)}$ is open in $\mathbb{R}^4$, and $\dim(\mathbb{R}^4 \setminus \mathbb{R}^{b_1(G)}) = 4$. This yields the theorem:

Theorem 5.4. *Given a noncommutative Lie group G of real dimension 4, the following hold:*

(1) *the space of left-invariant affinor metric structures with rank 0 radical (exact almost Kähler structures) on G is parametrized by the elements of the set $\mathbb{R}^4 \setminus \mathbb{R}^{b_1(G)} \times \mathbb{C}_-^2$ and has real dimension 8;*

(2) *the space of left-invariant affinor metric structures with rank 2 radical on G is parametrized by the elements of the set $\mathbb{R}^4 \setminus \mathbb{R}^{b_1(G)} \times \mathbb{C}_-$ and has real dimension 6.*

REMARK 5.5. All left-invariant 1-forms are closed on every commutative Lie group; consequently, no commutative Lie group admits a left-invariant affinor metric structure.

Suppose that (α, Φ, g) is a left-invariant affinor metric structure on a connected Lie group G of real dimension 4, with $\operatorname{rank}(\operatorname{rad}\alpha) = 2$, and denote by D the work bundle of this affinor metric structure. If D is a subalgebra in $\mathfrak{g}$, then D either is a 2-dimensional abelian subalgebra or is isomorphic to the Lie algebra $\mathfrak{e}(1)$ of the group $\mathrm{E}(1)$ of affine transformations of the real line $\mathbb{R}$. If D is an abelian subalgebra then $d\alpha|_D = 0$, which contradicts the condition that the 1-form α is nondegenerate on the work bundle. Consequently, $D \cong \mathfrak{e}(1)$. Then we can choose in D two left-invariant vector fields e_1 and e_2 with $[e_1, e_2] = e_2$. Since every almost complex structure on a manifold of real dimension 2 is integrable (see [7, Vol. 2]), the restriction of the affinor Φ on the subgroup $\mathrm{E}(1)$ is a complex structure on the latter, while $d\alpha_\Phi$ is a Kähler metric on it. This yields the following result:

Theorem 5.6. *A left-invariant affinor metric structure* (α, Φ, g) *with* $\operatorname{rank}(\operatorname{rad}\alpha) = 2$ *on a connected Lie group of real dimension* 4 *with work bundle D is sub-Kähler if and only if D contains two linearly independent left-invariant vector fields e_1 and e_2 with* $[e_1, e_2] = e_2$.

6. Invariant Affinor Metric Structures on the Homogeneous Spaces of Dimension 5

Consider a Riemannian homogeneous space M of real dimension 5, a group G of isometries acting on M transitively and faithfully, and the isotropy subgroup H of a source point o. By Theorem 2.6, the rank of every G-invariant 1-form on M which is not closed must be 1 or 3. The affinor metric structures on a 5-dimensional homogeneous space M are precisely the almost contact metric structures on M. Theorem 2.12 and Remark 2.13 imply the following statement:

Theorem 6.1. *Consider a Riemannian homogeneous space* $M = G/H$ *of dimension* 5 *and the orthogonal complement* $\mathfrak{p}$ *to the isotropy subalgebra* $\mathfrak{h}$, *on assuming that* Ad_H *acts irreducibly on* $\mathfrak{p}$. *Then M admits no G-invariant affinor* (*almost contact*) *metric structures.*

Theorem 6.2. *If a Riemannian homogeneous space* $M = G/H$ *of dimension* 5 *is isometric to the homogeneous space* $M_1 \times L$, *where* M_1 *is a simply-connected Riemannian homogeneous space of dimension* 4 *not isometric to a Lie group with a left-invariant Riemannian metric, and L is a homogeneous space of dimension* 1, *then M admits no G-invariant affinor* (*almost contact*) *metric structures.*

PROOF. Since $G = G_1 \times G_2$, where G_1 is a Lie group acting transitively on M_1 and G_2 is a Lie group acting transitively on L, by Theorems 5.1 and 5.2 every G-invariant 1-form α on M which is not closed amounts to $\alpha = \lambda\xi^*$, where ξ is a basis vector field on L with $\xi^*|_M = 0$ and $\lambda \in \mathbb{R}$. Hence,

$$d\alpha = \lambda d\xi^* = 0;$$

i.e., every G-invariant 1-form on M is closed. Consequently, M admits no G-invariant affinor metric structures.

6.1. The 5-dimensional sphere. Regard the 5-dimensional sphere S^5 as a subset of $\mathbb{C}^3$:

$$\{(z_1, z_2, z_3) : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}.$$

Define the action of the group $S^1 = \{e^{it} : t \in [0, 2\pi]\}$ on S^5 as

$$e^{it}(z) = (e^{it}z_1, e^{it}z_2, e^{it}z_3).$$

A tangent vector field to the orbits of this action is nowhere vanishing; consequently, S^5 has Euler class zero (see [5]). If we treat S^5 as the homogeneous space $\mathrm{SO}(6)/\mathrm{SO}(5)$, then S^5 is a Riemannian homogeneous space, and Theorem 6.1 shows that S^5 admits no $\mathrm{SO}(6)$-invariant affinor metric structures.

Furthermore, S^5 is the total space of the Hopf bundle $S^5 \to \mathbb{C}P^2$ with fiber S^1. In this case the group S^1 acts on S^5 not transitively. Take some connection Q on S^5 with connection form ω. The definition of connection form on a principal bundle (see [7, Vol. 1]) implies that Ω is an S^1-invariant 1-form on S^5. Denote by Ω the curvature form of Q. The structure equation

$$d\omega = \Omega + \frac{1}{2}\,[\omega, \omega]$$

yields $d\omega = \Omega$. The connection form ω is closed if and only if Q is flat. Suppose that Q is not flat. Denote by g the Riemannian metric induced by the embedding of S^5 into $\mathbb{C}^3$, and by Φ, the affinor associated to ω. Then (ω, Φ, g) is an S^1-invariant affinor (almost contact) metric structure on S^5. If the restriction of the curvature form Ω to $\mathbb{C}P^2$ is a symplectic form then (ω, Φ, g) is an S^1-invariant contact metric structure on S^5.

6.2. Products of the 3-dimensional sphere. Consider $M = S^3 \times R$, where R is a Riemannian homogeneous space of real dimension 2, and a group G of isometries acting on M transitively and faithfully. If α is a G-invariant 1-form on M with $TR \subset \ker \alpha$ then all vector fields X and Y on R satisfy $\alpha([X,Y]) = 0$. Since for every vector field X on R and every vector field Y on M we have $[X,Y] = 0$ and the 1-form α is G-invariant, it follows that $d\alpha(X,Y) = 0$ for every vector field X on R and every vector field Y on M. Since the 3-dimensional sphere S^3 is isometric to the group SU(2), while the Lie algebra of this group is isomorphic to $\mathbb{R}^3$ with the cross product as multiplication, there exist three linearly independent left-invariant 1-forms with rank 1 radicals on SU(2). On assuming that these 1-forms vanish identically on R and choosing as the metric the sum of metrics on S^3 and R, we see that M admits a G-invariant affinor metric structure with rank 3 radical. This implies that each left-invariant contact metric structure on SU(2) determines a G-invariant affinor metric structure with rank 3 radical on the homogeneous spaces $S^3 \times S^2$, $S^3 \times \mathbb{R}^2$, $S^3 \times T^2$, where T^2 is a 2-dimensional flat torus.

6.3. The Lie groups of dimension 5. Consider a 5-dimensional real Lie group G and take a left-invariant 1-form α on G which is not closed. By Theorem 2.6, in some basis of left-invariant vector fields $e_1, \dots, e_5$ the exterior 2-form $d\alpha$ becomes either

$$d\alpha = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* \quad \text{or} \quad d\alpha = e_1^* \wedge e_2^*.$$

In both cases the left-invariant affinor associated to the 1-form α in the canonical basis becomes $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, where A is a 4×4 or 2×2 nonzero block.

Denote by $b_1(G)$ the first Betti number of G. The set of all left-invariant 1-forms on G which are not closed is parametrized by the elements of the set $\mathbb{R}^5 \setminus \mathbb{R}^{b_1(G)}$. Introducing the sets $\mathbb{C}_-$ and $\mathbb{C}_-^2$ and arguing as in Section 5.3, we obtain the description for the left-invariant affinor metric structures on G:

Theorem 6.3. *Given a noncommutative Lie group G of real dimension 5, the following hold:*

(1) *the space of left-invariant affinor metric structures with rank 1 radical on G is parametrized by the elements of the set $\mathbb{R}^5 \setminus \mathbb{R}^{b_1(G)} \times \mathbb{C}_-^2$ and has real dimension 9.*

(2) *the space of left-invariant affinor metric structures with rank 3 radical on G is parametrized by the elements of the set $\mathbb{R}^5 \setminus \mathbb{R}^{b_1(G)} \times \mathbb{C}_-$ and has real dimension 7.*

A strict affinor metric structure with rank 1 radical is a contact metric structure. The left-invariant contact metric structures on the Lie groups of dimension 5 are fully described in [8].

6.4. The Lie groups of arbitrary dimension. Consider a Lie group G of real dimension $n \geq 3$ and take a left-invariant 1-form α on G, which is not closed, with $\operatorname{rank}(\operatorname{rad}\alpha) = r \leq n-2$. By Corollary 2.7, there exists an integer k with $n - r = 2k$. Let $\mathbb{C}_-^k$ stand for the following set: $\{(z_1, \dots, z_k) \in \mathbb{C}^k : \operatorname{Im} z_1 < 0, \dots, \operatorname{Im} z_k < 0\}$. Generalizing Theorems 5.4 and 6.3 to the Lie groups of arbitrary dimension, we can prove the following result.

Theorem 6.4. *Suppose that G is a noncommutative Lie group of real dimension $n \geq 3$ with the first Betti number $b_1(G)$. Then the space of left-invariant affinor metric structures with rank r radical on G is parametrized by the elements of the set $\mathbb{R}^n \setminus \mathbb{R}^{b_1(G)} \times \mathbb{C}_-^k$, $k = \frac{n-r}{2}$, and has real dimension $n + 2k$.*

Take a nonzero left-invariant 1-form α on G and a left-invariant Riemannian metric β on G. By Riesz's Theorem (see [6]) the group G admits a left-invariant vector field ξ so that $\alpha(X) = \beta(\xi, X)$ for $X \in \mathfrak{g}$. All left-invariant vector fields X and Y on G satisfy

$$d\alpha(X,Y) = -\frac{1}{2}\alpha([X,Y]) = -\frac{1}{2}\beta(\xi,[X,Y]).$$

If G is a semisimple Lie group then the 1-form α cannot be closed because $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. This yields the following corollary:

Corollary 6.5. *If G is a semisimple Lie group of real dimension $n \geq 3$ then the space of left-invariant affinor metric structures with rank r radical on G is parametrized by the elements of the set $\mathbb{R}^n \setminus \{0\} \times \mathbb{C}^k_-$, $k = \frac{n-r}{2}$, and has real dimension $2n - r$.*

Remark 6.6. Since the first derived ideal of the radical of every left-invariant affinor metric structure on a Lie group is a subalgebra orthogonal to the characteristic vector field, no left-invariant affinor metric structure with semisimple radical can be strict.

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